

Solve $2^x = x^4$ and its general form

By Trinh, Spring-break 2025

General form, solve $A^x = x^B$ ($A > 0$ and $x \neq 0$)

Goal: Convert to form $k^{\frac{1}{k}} = x^{\frac{1}{x}}$ \rightarrow Solution $x = k$

Power both sides to $\frac{1}{Bx} \rightarrow (A^x = x^B)^{\frac{1}{Bx}} \rightarrow A^{\frac{1}{B}} = x^{\frac{1}{x}}$

Work on the left side to convert it to form $k^{\frac{1}{k}} = A^{\frac{1}{B}}$

We want to obtain a variable X such that $A^X = BX = k$

Sub work

Think of equation in the form $A^X = BX$

Multiply both sides by $-A^{-X} \rightarrow -A^{-X}(A^X = BX) \rightarrow -1 = -BXA^{-X}$

Rewrite $A = e^{\ln(A)}$ We have $-1 = -BXe^{-X\ln(A)}$

Multiply $\frac{\ln(A)}{B}$ both sides $\rightarrow \frac{\ln(A)}{B}(-1) = \left(\frac{\ln(A)}{B}\right)(-BXe^{-X\ln(A)})$

$$\frac{-\ln(A)}{B} = -X\ln(A)e^{-X\ln(A)}$$

Apply [Lambert function](#) (product logarithm) both sides

$$\begin{aligned} W_0(xe^x) &= x & \text{for } x \geq -1, \\ W_{-1}(xe^x) &= x & \text{for } x \leq -1. \end{aligned}$$

$$W\left(\frac{-\ln(A)}{B}\right) = W(-X\ln(A)e^{-X\ln(A)})$$

$$\text{Or } W\left(\frac{-\ln(A)}{B}\right) = -X\ln(A) \rightarrow X = \frac{W\left(\frac{-\ln(A)}{B}\right)}{-\ln(A)}$$

Back to main work, since

$$A^x = Bx = k \rightarrow k = A^x \text{ or } k = BX \rightarrow k = B \frac{W\left(\frac{-\ln(A)}{B}\right)}{-\ln(A)}$$

$$\text{Now we have the solution for our problem } x = k = B \frac{W\left(\frac{-\ln(A)}{B}\right)}{-\ln(A)}$$

Don't be confused between x in the main solution with X in sub work

Finally, back to our original, solve $A^x = x^B$ ($A > 0$ and $x \neq 0$)

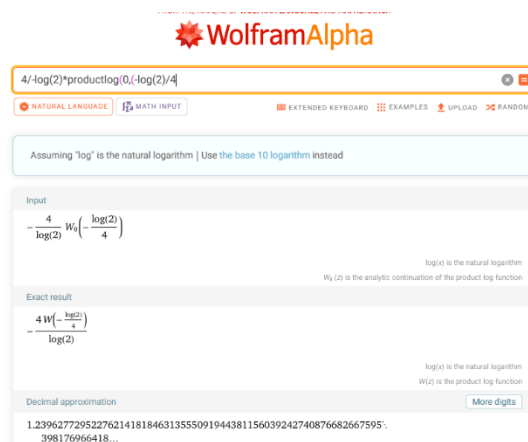
$$\text{Solution } k = x = B \frac{W\left(\frac{-\ln(A)}{B}\right)}{-\ln(A)}$$

Using either principal branch W_0 or secondary branch W_{-1}

or both to obtain the real solution(s)

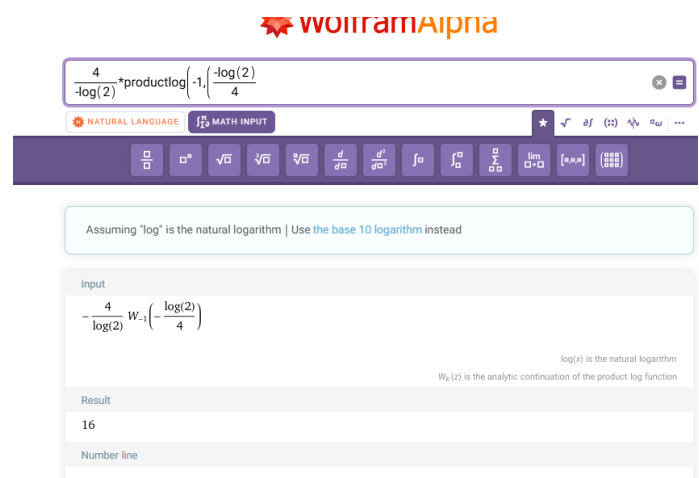
$$\text{Solve } 2^x = x^4 \quad A = 2 \quad B = 4 \quad \rightarrow x = B \frac{W\left(\frac{-\ln(A)}{B}\right)}{-\ln(A)} = 4 \frac{W\left(\frac{-\ln(2)}{4}\right)}{-\ln(2)}$$

[Link to solution 1](#) $x = 1.2396$



WolframAlpha interface showing the input $4/\log(2)*\text{productlog}(0,-\log(2)/4)$. The interface includes tabs for NATURAL LANGUAGE and MATH INPUT, and buttons for EXTENDED KEYBOARD, EXAMPLES, UPLOAD, and RANDOM. A message states: "Assuming 'log' is the natural logarithm | Use the base 10 logarithm instead". The input field shows $-\frac{4}{\log(2)} W_0\left(-\frac{\log(2)}{4}\right)$. The exact result is $-\frac{4 W_0\left(-\frac{\log(2)}{4}\right)}{\log(2)}$. The decimal approximation is $1.2396277295227621418184631355091944381156039242740876682667595$.

[Link to solution 2](#) $x = 16$



WolframAlpha interface showing the input $4/\log(2)*\text{productlog}(-1,-\log(2)/4)$. The interface includes tabs for NATURAL LANGUAGE and MATH INPUT, and buttons for EXTENDED KEYBOARD, EXAMPLES, UPLOAD, and RANDOM. A message states: "Assuming 'log' is the natural logarithm | Use the base 10 logarithm instead". The input field shows $-\frac{4}{\log(2)} W_{-1}\left(-\frac{\log(2)}{4}\right)$. The result is 16 . The number line shows a single point at 16.

I wrote a solution checking with productlog function embedded in demos here.

(Just google desmos productlog function)

<https://www.desmos.com/calculator/q2cjgfmcxl>