Solve $2^x = x^4$ and its general form

By Trinh, Spring-break 2025

General form, solve $A^x = x^B$ (A > 0 and $x \neq 0$)

Goal: Convert to form $k^{\frac{1}{k}} = x^{\frac{1}{x}}$ → Solution $x = k$

Power both sides to $\frac{1}{Bx}$ → $(A^x = x^B)^{\frac{1}{Bx}}$ → $A^{\frac{1}{B}} = x^{\frac{1}{x}}$

Work on the left side to convert it to form $k^{\frac{1}{k}} = A^{\frac{1}{B}}$

We want to obtain a variable X such that $A^X = BX = k$

Sub work

Think of equation in the form $A^X = BX$

Multiply both sides by $-A^{-X}$ → $-A^{-X}(A^X = BX)$ → $-1 = -BXA^{-X}$

Rewrite $A = e^{\ln(A)}$ We have $-1 = -BXe^{-Xln(A)}$

Multiply $\frac{\ln(A)}{B}$ both sides → $\frac{\ln(A)}{B}(-1) = \left(\frac{\ln(A)}{B}\right)\left(-BXe^{-Xln(A)}\right)$

$$\frac{-\ln(A)}{B} = -X\ln(A)e^{-Xln(A)}$$

Apply Lambert function (product logarithm) both sides

$$W_0(xe^x) = x \qquad \text{for } x \geq -1,$$
$$W_{-1}(xe^x) = x \qquad \text{for } x \leq -1.$$

$$W\left(\frac{-\ln(A)}{B}\right) = W\left(-Xln(A)e^{-Xln(A)}\right)$$

Or $W\left(\frac{-\ln(A)}{B}\right) = -Xln(A)$ → $X = \frac{W\left(\frac{-\ln(A)}{B}\right)}{-\ln(A)}$

Back to main work, since

$A^x = Bx = k$ → $k = A^X$ or $k = BX$ → $k = B\frac{W\left(\frac{-\ln(A)}{B}\right)}{-\ln(A)}$

Now we have the solution for our problem $x = k = B\frac{W\left(\frac{-\ln(A)}{B}\right)}{-\ln(A)}$

Don't be confused between x in the main solution with X in sub work

Finally, back to our original, solve $A^x = x^B$ (A > 0 and $x \neq 0$)

Solution $k = \boldsymbol{x} = \boldsymbol{B}\dfrac{\boldsymbol{W}\left(\frac{-\ln(A)}{B}\right)}{-\ln(A)}$

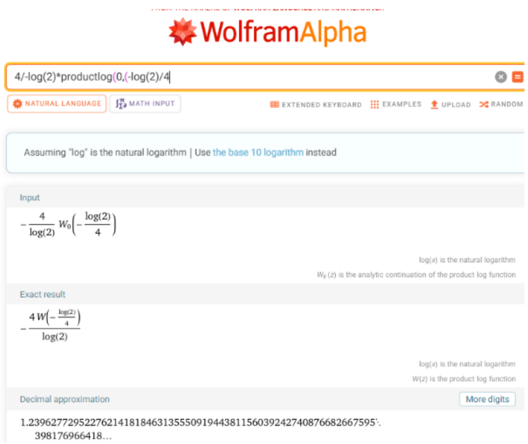
Using either principal branch W_0 or secondary branch W_{-1}

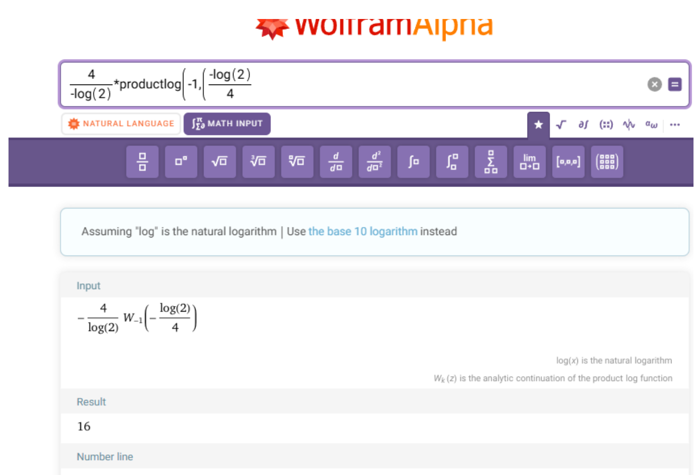
or both to obtain the real solution(s)

Solve $2^x = x^4$ A = 2 B= 4 → $x = B\dfrac{W\left(\frac{-\ln(A)}{B}\right)}{-\ln(A)} = 4\dfrac{W\left(\frac{-\ln(2)}{4}\right)}{-\ln(2)}$

Link to solution 1 $x = 1.2396$

Link to solution 2 $x = 16$





I wrote a solution checking with productlog function embedded in demos here.

(Just google desmos productlog function)

https://www.desmos.com/calculator/q2cjgfmcxl